Thermal conductance in a spin-boson model: Cotunneling and low temperature properties

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Bosonic thermal transport through a two-level system is analyzed at temperatures below and comparable to the two-level energy splitting. It is shown that in the low-temperature regime transport is dominated by correlated two-boson processes analogous to electron cotunneling in quantum dots under Coulomb blockade. We present a detailed analysis of the sequential-cotunneling crossover and obtain essentially an analytic description of the transport problem. Perturbative analysis is complemented by employing scaling properties of the Ohmic spin-boson model, allowing us to extract an anomalous low temperature scaling of thermal conductance.

I. INTRODUCTION

The spin-boson model describes the interaction between bosonic modes and a two-level quantum system.[1] It has been employed in a wide variety of phenomena exhibiting dissipation and loss of quantum coherence. Lately the model has found applications in a nonequilibrium setting where two bosonic reservoirs with different temperatures couple through the two-level system (TLS).[2–9] These considerations are motivated by efforts to understand thermal properties of artificial molecular and nanoscale structures. The spin-boson model is applicable to a generic situation where bosonic fluctuations couple through a nonlinear region at low enough temperatures so that a truncation to the lowest two energy levels is sufficient. The spin-boson model can be regarded as a bosonic counterpart of the Anderson model which describes dynamics of a single spin-degenerate electron orbital coupled to leads. Although these model systems provide a highly simplified picture of real physical systems, they often capture essential properties successfully.

The spin-boson model is interesting from the point of view of *heattronics*, the manipulation and control of heat in artificial nanostructures.[10,11] Nonlinear elements, a two-level system being the simplest, with tunable parameters are basic building blocks for manipulation of heat flow. Possible realizations of the spin-boson thermal transport are, for example, molecular junctions coupling two phonon reservoirs or electric circuits coupled through a superconducting qubit,[12] see Fig. 1. In the latter example the qubit can be manipulated by external fields allowing a flexible control of its properties and enabling the investigation of different transport regimes.

In this paper we present a perturbative theory of thermal transport in the spin-boson model for arbitrary bath temperatures. At temperatures above or comparable to the TLS energy splitting the reservoirs can excite the system and transport is dominated by incoherent sequential emission and absorbtion processes. Except for one numerical simulation,[9] all previous investigations of spin-boson heat transport[2–8] have considered only this
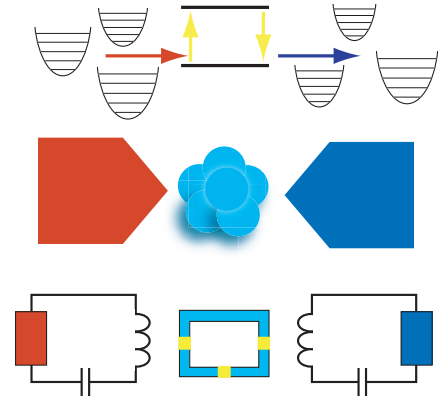

FIG. 1: (Top) Schematic picture of thermal transport in a two-reservoir spin-boson model. A two-level system mediates energy transfer between two harmonic oscillator baths held at different temperatures. Possible experimental realizations: (center) phonons in two bulk leads couple through a nonlinear molecular junction, (bottom) electromagnetic fluctuations of two linear circuits couple through a superconducting qubit.

regime. But when the bath temperatures decrease below the two-level splitting it is no longer possible to excite the mediating system and sequential processes are blocked. In this case one needs to take into account coherent two-boson processes which correspond to virtual excitation of the two-level system. The qualitative picture is the same as in electron transport in quantum dots under Coulomb blockade. At low enough bias voltage the charging energy blocks exponentially the sequential tunneling processes where individual electrons hop in and out of the dot incoherently.[13] The dominant transport processes then consist of coherent two-electron cotunneling processes where the population of the dot remains the same after the process.[14] In analogy to electron transport, we use here the word "cotunneling" for the virtual two-boson processes giving rise to the low temperature heat conductance. By calculating two-boson cotunneling

rates we obtain an analytic description for thermal transport below the sequential-cotunneling crossover. Our results are valid for weak bath coupling or for an arbitrary coupling strength when the noncommuting part of the two-level Hamiltonian and coupling operators is small. We present a detailed analysis of the crossover region and discuss the applicability of the sequential tunneling approximation. By employing the scaling equations of the Ohmic spin-boson model we can take into account renormalization of the spin-boson parameters. This allows us to complement the perturbative results by scaling effects and extract an anomalous temperature scaling of cotunneling thermal conductance.

The paper is organized in the following way. In Section II we introduce the studied model in detail. In Section III we calculate the cotunneling transition rates of relevant two-boson processes. We present a general formula for heat current from which one can obtain an analytic expressions for the low-temperature and high-temperature limits and study the crossover behavior. In Section IV we use the scaling equations of the Ohmic spin-boson model to calculate temperature scaling of the cotunneling thermal conductance and in Section V we conclude our analysis.

II. MODEL

We model our system with the usual spin-boson Hamiltonian, separating the environmental oscillators into left (L) and right (R) baths:

$$H = H_0 + H_T, \tag{1}$$

$$H_0 = \tfrac{1}{2}\epsilon\sigma_z + \tfrac{1}{2}\Delta\sigma_x + \sum_{i\in L,R} \omega_i(a_i^\dagger a_i + \tfrac{1}{2}), \tag{2}$$

$$H_T = \sigma_z \sum_{i\in L,R} C_i(a_i + a_i^\dagger). \tag{3}$$

The TLS is described by level detuning ϵ and transition amplitude Δ, while the baths are characterized by their spectral functions $\chi_K(\omega) = 2\pi\sum_{i\in K} C_i^2 \delta(\omega - \omega_i)$, $K = L, R$. Equations (1)-(3) constitute the most generic two-bath spin-boson system.[1]

It is convenient to diagonalize also the TLS part of H_0 with a rotation of the pseudospin by an angle θ in the x-z plane, with $\tan\theta = \Delta/\epsilon$. In this new basis we have

$$H_0 = \tfrac{1}{2}\omega_0\sigma_z + \sum_{i\in L,R} \omega_i(a_i^\dagger a_i + \tfrac{1}{2}), \tag{4}$$

$$H_T = (\sigma_z\cos\theta - \sigma_x\sin\theta)\sum_{i\in L,R} C_i(a_i + a_i^\dagger), \tag{5}$$

where $\omega_0 = \sqrt{\epsilon^2 + \Delta^2}$. It should be noted that the validity of our perturbation expansion requires that the TLS level broadenings Γ_0 and Γ_1, given by Eqs. (16) and (17) below, are small compared to the level separation ω_0. Since $\Gamma_{0,1}$ contain the couplings C_i as well as $\sin\theta$, our results are valid either for weak coupling or when the noncommuting part of the coupling is small ($\theta \ll 1$).

III. FORMULA FOR THE CURRENT

Our calculation of heat current between the baths is based on the generalized Fermi Golden Rule.[15] Given two eigenstates $|i\rangle$ and $|f\rangle$ of H_0, the perturbation H_T induces transitions between them at a rate

$$\Gamma_{i\to f} = 2\pi\big|\langle f|T|i\rangle\big|^2 \delta(E_i - E_f), \tag{6}$$

where E_i and E_f are energies of the two states, and the scattering operator T is

$$T = H_T + H_T G_0 H_T + H_T G_0 H_T G_0 H_T + \dots \tag{7}$$

Here G_0 is the retarded propagator for H_0 at energy E_i,

$$G_0 = \frac{1}{E_i - H_0 + i\eta}. \tag{8}$$

We will consider only those processes where one quantum of energy is transported between the baths, i.e., the initial and final states are related as

$$|f\rangle = \frac{1}{\sqrt{(N_{K'} + 1)N_K}} a_{K'}^\dagger a_K |i\rangle, \tag{9}$$

where K and K' refer to arbitrary oscillators in different baths and N_i are their occupation numbers. In principle the initial and final states of the TLS could be different but below we show that these inelastic processes vanish for two-boson transport. Restriction to Eq. (9) forbids processes with an odd number of bosons and therefore we only keep the even powers of H_T in Eq. (7), yielding[16]

$$T = H_T G H_T, \tag{10}$$

where G is solved from a Dyson equation as

$$G = \frac{1}{E_i - H_0 - \Sigma}, \tag{11}$$

with self-energy $\Sigma = H_T G_0 H_T$. We consider the self-energy to the lowest order and keep only the imaginary part which regularizes the divergence at ω_0. We will now show that in this case the off-diagonal elements of Σ can be neglected. Let us decompose the T matrix elements as

$$\langle f|T|i\rangle = \sum_{j,k,\sigma,\sigma'} \langle f|H_T|k\sigma'\rangle\langle k\sigma'|G|j\sigma\rangle\langle j\sigma|H_T|i\rangle, \tag{12}$$

where $|j\rangle$ and $|k\rangle$ are states in the bath subspace, and $|\sigma\rangle$ and $|\sigma'\rangle$ in the TLS subspace. Similarly we can write Σ as

$$\langle k\sigma'|\Sigma|j\sigma\rangle = \sum_m \left\{ \frac{\langle k\sigma'|H_T|m\sigma\rangle\langle m\sigma|H_T|j\sigma\rangle}{E_i - E_{m\sigma} + i\eta} + \frac{\langle k\sigma'|H_T|m\bar\sigma\rangle\langle m\bar\sigma|H_T|j\sigma\rangle}{E_i - E_{m\bar\sigma} + i\eta} \right\}, \tag{13}$$

where $|\bar\sigma\rangle$ is the state with the spin flipped with respect to $|\sigma\rangle$. As noted above, we will only be concerned with the

imaginary part of the self-energy, and for the first term on the RHS it is nonzero when $E_i = E_{m\sigma}$. On the other hand, to lowest order the self-energy is used to regularize the divergence of sequential energy transfer, and in that case the coupling terms H_T in Eq. (12) produce energy-conserving processes, i.e., $E_f = E_{k\sigma'}$ and $E_i = E_{j\sigma}$. Thus we have $E_{m\sigma} = E_{j\sigma}$, implying $|m\rangle = |j\rangle$ and $\langle m\sigma|H_T|j\sigma\rangle = 0$. The first term of Eq. (13) can therefore be neglected. For the second term a non-vanishing imaginary part requires $E_i = E_{m\bar{\sigma}}$, Eq. (6) gives $E_i = E_f$, and the sequential condition is $E_f = E_{k\sigma'}$, which combine to $E_{m\bar{\sigma}} = E_{k\sigma'}$. Therefore if $\sigma' = \bar{\sigma}$ we have $|m\rangle = |k\rangle$ and the second term also vanishes. Only the spin-diagonal elements $\sigma' = \sigma$ of the second term give a finite imaginary part. This also means that within our weak-coupling approximation only spin-flipping processes, proportional to σ_x in H_T, contribute to Σ.

Next we argue that the self-energy elements which are off-diagonal in the bath subspace can also be neglected. From Eq. (13) one can see that nonvanishing off-diagonal elements must have $|k\rangle = a_1^{(\dagger)} a_2^{(\dagger)} |j\rangle$, where $a_i^{(\dagger)}$ denotes either a creation or destruction operator for the baths. In this case the sum contains only two terms, with $|m\rangle = a_1^{(\dagger)}|j\rangle$ or $|m\rangle = a_2^{(\dagger)}|j\rangle$. In contrast, for diagonal elements $|j\rangle = |k\rangle$ the sum goes over all states of the form $|m\rangle = a_m^{(\dagger)}|j\rangle$. Since each term in the sum is proportional to the coupling, at least in our weak-coupling approximation the diagonal elements dominate and the off-diagonal ones can be discarded. Thus we obtain the expression

$$-\text{Im}\,\langle j\sigma|\Sigma|j\sigma\rangle = \pi \sum_m \left|\langle m\bar{\sigma}|H_T|j\sigma\rangle\right|^2 \delta(E_{j\sigma} - E_{m\bar{\sigma}})$$
$$\equiv \tfrac{1}{2}\Gamma_{j\sigma}, \tag{14}$$

that is, the imaginary part of the self-energy element is half of the Golden-Rule transition rate out of state $|j\sigma\rangle$. Collecting everything yields

$$\Gamma_{i\to f} = 2\pi \left| \sum_{j,\sigma} \frac{\langle f|H_T|j\sigma\rangle\langle j\sigma|H_T|i\rangle}{E_i - E_{j\sigma} + i\tfrac{1}{2}\Gamma_{j\sigma}} \right|^2 \delta(E_f - E_i). \tag{15}$$

Evaluating the sum over the intermediate states $|j\sigma\rangle$ is straightforward. We note that the calculation shows that even for cotunneling (with $E_i \neq E_{j\sigma}$) it is again only the spin-flipping terms, proportional to σ_x in H_T, which give a non-vanishing contribution. The coupling term proportional to σ_z is therefore completely irrelevant for the two-boson weak-coupling current. This is intuitively plausible since such a term only produces a small shift in the system energy levels.

Next we replace the lifetime $\Gamma_{j\sigma}^{-1}$ in Eq. (15) with the lifetime of a state where the TLS subspace is in state $|\sigma\rangle$ but the bath subspaces are in thermal states. Then $\Gamma_{j\sigma}$ becomes independent of the bath state $|j\rangle$, and we call the two possible values Γ_0 and Γ_1, corresponding to ground state and excited TLS, respectively. A short calculation

gives

$$\Gamma_0 = \Gamma_L e^{-\omega_0/T_L} + \Gamma_R e^{-\omega_0/T_R}, \tag{16}$$
$$\Gamma_1 = \Gamma_L + \Gamma_R, \tag{17}$$
$$\Gamma_K = \sin^2\theta\,\chi_K(\omega_0)\big[n_K(\omega_0) + 1\big], \;\; K = L, R. \tag{18}$$

Here Γ_K is the decay rate of the excited TLS into bath K with temperature T_K and corresponding Bose distribution n_K.

To derive a formula for the heat current we first write down an expression for heat absorbed from the left bath and emitted to the right bath, with σ being the initial (and final) state of the TLS

$$J_{L\to R}^{(\sigma)} = \sum_{i,f} \omega_{f_L} \Gamma_{i\to f} W_i, \tag{19}$$

where the sum is over all possible initial states of the reservoirs (with a thermal distribution W_i), and all final states of the form (9) with $K = f_L$ an arbitrary oscillator in the left bath and $K' = f_R$ in the right bath. Energy conservation requires that $\omega_{f_L} = \omega_{f_R}$. Current in the other direction, $J_{R\to L}^{(\sigma)}$, is defined analogously, and the net heat flow is $J^{(\sigma)} = J_{L\to R}^{(\sigma)} - J_{R\to L}^{(\sigma)}$ which can be evaluated as

$$J^{(\sigma)} = \frac{\sin^4\theta}{2\pi} \int_0^\infty d\omega\,\omega\chi_L(\omega)\chi_R(\omega)\big[n_L(\omega) - n_R(\omega)\big]$$
$$\times \left| \frac{1}{\omega - \omega_0 \pm \tfrac{i}{2}\Gamma_{\bar\sigma}} - \frac{1}{\omega + \omega_0 \mp \tfrac{i}{2}\Gamma_{\bar\sigma}} \right|^2, \tag{20}$$

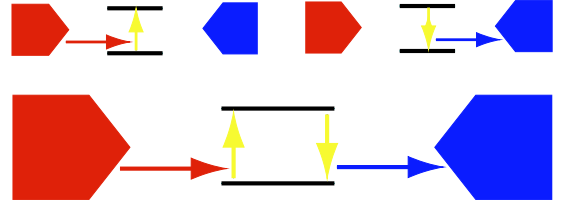where upper signs refer to $\sigma = 0$, lower signs to $\sigma = 1$.

FIG. 2: (Top) At temperatures comparable to the two-level energy splitting transport is dominated by incoherent sequential processes. (Bottom) At low temperatures it is no longer possible to excite the two-level system and transport is dominated by coherent two-boson processes creating virtual excitations.

We still have to combine the two partial currents $J^{(\sigma)}$ to obtain the full heat flow through the system. First note that under steady-state conditions the probability for the TLS to be in ground state is $P_0 = \Gamma_1/(\Gamma_0 + \Gamma_1)$, and for the excited state $P_1 = 1 - P_0$. Let us then split the currents into sequential and cotunneling contributions,[17] $J^{(\sigma)} = J_{\text{seq}}^{(\sigma)} + J_{\text{cot}}^{(\sigma)}$, see also Fig. 2. For sequential processes the intermediate state has the TLS spin flipped, for cotunneling processes this flipping is only virtual and

does not contribute to $P_{0/1}$. Now, for sequential transport, one current-carrying process consists of two incoherent tunneling events, and if we start with, say, the TLS in ground state, then the intermediate state is excited. But this intermediate state can equally well be seen as the initial state of another current-carrying process. Because of this overlap, one must be careful to avoid double counting, and as further elucidated in Fig. 3, the full sequential current is $J_{\text{seq}} = P_0 J_{\text{seq}}^{(0)} = P_1 J_{\text{seq}}^{(1)}$.
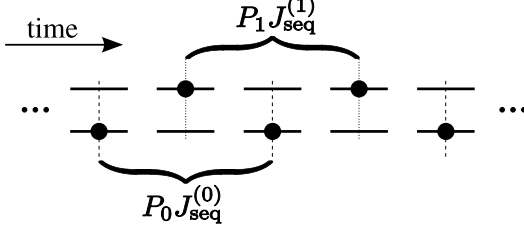

FIG. 3: Evolution of the TLS state due to sequential tunneling events. The current carried by these tunneling events can be calculated in two different ways. On one hand, we can start with the TLS in ground state, and then one current-carrying process flips the state twice, and the next process again starts from the ground state. The average current transported by these processes is $J_{\text{seq}} = P_0 J_{\text{seq}}^{(0)}$. On the other hand, we can describe the same sequence of events by starting with the TLS in the excited state, giving $J_{\text{seq}} = P_1 J_{\text{seq}}^{(1)}$. Thus the two expressions for sequential current are not additive but alternatives to one another.

On the other hand, cotunneling events are non-overlapping and therefore they contribute additively, $J_{\text{cot}} = P_0 J_{\text{cot}}^{(0)} + P_1 J_{\text{cot}}^{(1)}$. Total heat current through the system, $J = J_{\text{seq}} + J_{\text{cot}}$, is therefore

$$J = P_0 J^{(0)} + P_1 J^{(1)} - J_{\text{seq}}. \qquad (21)$$

The sequential current can be calculated by taking the difference of the forward and backward Golden Rule rates at either junction, e.g., for the left junction,

$$J_{\text{seq}} = \omega_0 (P_0 \Gamma_L e^{-\omega_0/T_L} - P_1 \Gamma_L). \qquad (22)$$

We can still simplify Eq. (21). As will be shown below, at high temperatures sequential tunneling dominates and all three terms on the RHS of Eq. (21) are equal in magnitude and thus the last two terms cancel. They can also be neglected at low temperatures since then P_1 and J_{seq} are exponentially suppressed. For these limits we therefore have

$$J = P_0 J^{(0)}. \qquad (23)$$

Numerical calculations suggest that this expression is actually valid within a few precent at all temperatures. Equation of the same form has been previously used by Flensberg[16] in the context of electron cotunneling.

A. High-temperature limit

Next we will derive the high and low-temperature limits for Eq. (23), corresponding to sequential and cotunneling, respectively. At high temperatures resonant energy transfer at $\omega = \omega_0$ dominates, and therefore the second energy denominator in Eq. (20) can be dropped. Contribution from the resonance peak can be calculated by taking the limit $\Gamma_{0,1} \to 0$, implying $|\omega - \omega_0 \pm \frac{i}{2}\Gamma|^{-2} \to \frac{2\pi}{\Gamma}\delta(\omega - \omega_0)$. The current is then

$$J = \frac{\sin^2 \theta \, \omega_0 \chi_L(\omega_0) \chi_R(\omega_0) \big[n_L(\omega_0) - n_R(\omega_0) \big]}{\chi_L(\omega_0) \big[2n_L(\omega_0) + 1 \big] + \chi_R(\omega_0) \big[2n_R(\omega_0) + 1 \big]}. \qquad (24)$$

This expression is equivalent to the sequential tunneling current calculated from Eq. (22), and to the result derived by Segal and Nitzan.[2]

The processes leading to Eq. (24) correspond to sequential absorption-emission of bosons as depicted in Fig. 2. In this limit the heat current is sensitive only to the value of the bath spectral functions at resonance. At temperatures $T_{L,R} \ll \omega_0$ expression (24) becomes exponentially small, signalling the breakdown of the sequential tunneling approximation.

B. Low-temperature limit

In the low-temperature limit, $T_{L,R} \ll \omega_0$ we can ignore the ω-dependence of the energy denominators in Eq. (20). In addition, the excitation of the TLS is exponentially suppressed, so that $J = J^{(0)}$. Then the general expression reduces to

$$J = \frac{2 \sin^4 \theta}{\pi \omega_0^2} \int_0^\infty d\omega \, \omega \chi_L(\omega) \chi_R(\omega) \big[n_L(\omega) - n_R(\omega) \big]. \quad (25)$$

Transport is dominated by two-boson cotunneling processes during which one boson is absorbed and one emitted as in Fig. 2.

Let us remark that both Eqs. (24) and (25) can be cast in Landauer form, $J = \int d\omega \, \omega \mathcal{T}(\omega) \big[n_L(\omega) - n_R(\omega) \big]$, where $\mathcal{T}(\omega)$ is the transmission function. An important difference between the two cases is that in the sequential regime $\mathcal{T}$ depends on the bath temperatures while for cotunneling the transmission is temperature-independent. One consequence is that the cotunneling current is symmetric with respect to the reversal of the temperature gradient, and therefore in contrast to the high-temperature case,[2] the spin-boson system cannot act as a thermal rectifier at low temperatures.

For concreteness we calculate the cotunneling current in the case where the bath spectral densities have a power-law form $\chi_K(\omega) = A_K \omega^{s_K}$, with $0 < s_K < 1$ corresponding to sub-Ohmic, $s_K = 1$ to Ohmic, and $s_K > 1$ to super-Ohmic environment. Then we have

$$J = \frac{2 A_L A_R}{\pi \omega_0^2} \sin^4 \theta \, \Gamma(\nu) \zeta(\nu) \, (T_L^\nu - T_R^\nu), \qquad (26)$$

where $\nu = 2 + s_L + s_R$, and Γ and ζ are the gamma and zeta function, respectively. It is interesting to note that the temperature dependence of the current is sensitive only to the sum of the exponents $s_L + s_R$ and a precise nature of the individual baths is irrelevant.

C. Crossover

In Fig. 4 we plot the numerical solution of Eq. (23) together with the limiting forms given by Eqs. (24) and (26). We see that the crossover between sequential tunneling and cotunneling is quite sharp, and the two analytical formulas are very well applicable outside a narrow transition region. By patching together the two limiting expressions one achieves an accurate analytical description of the transport problem at all temperatures. The crossover temperature depends on the system parameters logarithmically and it is generally of the order of $0.1\,\omega_0$, below which the naive Golden-Rule treatment becomes invalid.
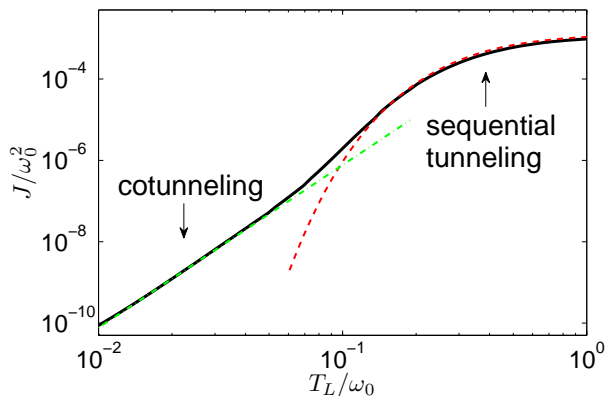

FIG. 4: Current through the system as a function of left-bath temperature T_L. Solid line is the numerical solution of Eq. (23), dashed line is the high-temperature approximation from Eq. (24), and dash-dotted line is the low-temperature approximation from Eq. (26). Both baths are ohmic, $s_L = s_R = 1$, with equal coupling strengths $A_L = A_R = 0.1$, and $\sin\theta = 1$. Right-bath temperature is $T_R = 0.95\,T_L$.

IV. LOW TEMPERATURE PROPERTIES FROM RENORMALIZATION GROUP

In the linear-response limit, $T_R \to T_L \equiv T$, Eq. (26) shows that in the cotunneling regime thermal conductance G depends on the temperature as $G \propto T^{1+s_L+s_R}$. This result was derived using perturbation theory and as such relies on an assumption of weak coupling between the TLS an the baths. In principle it is possible to calculate higher orders of perturbation theory and find corrections but this becomes soon unfeasible. Instead of carrying out a more complete perturbative calculation, in this section we use a renormalization group (RG) arguments to improve our analysis and find correction to the temperature exponent of $G(T)$.

To set up the RG framework, we introduce a cutoff frequency ω_c above which the bath response functions $\chi_{L,R}$ vanish. The initial value of ω_c is irrelevant for our purposes. We then perform the RG step by lowering the cutoff and integrating out all bath oscillators with frequencies above the new cutoff. This system is again described by a Hamiltonian of the form (1) but with a new set of parameters. The process is continued until ω_c is equal to the temperature T. This way the system parameters, which we have so far treated as constants, become effectively temperature dependent and give an extra contribution to $G(T)$.
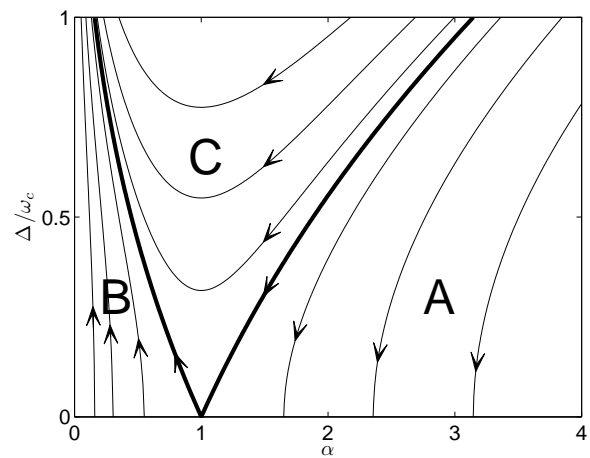

FIG. 5: Renormalization group flow of the parameters α and Δ/ω_c in the Ohmic spin-boson model, as given by Eqs. (27) and (28). Two separatrices (thick lines) divide the flows into three qualitatively different regimes.

The actual RG calculation is performed by the well-known Coulomb gas mapping.[1] This procedure is only available for Ohmic baths, and therefore for the rest of this section we set $s_L = s_R = 1$. The partition function is transformed to a form which is identical to the anisotropic Kondo model after the Yuval-Anderson mapping.[18] The renormalization procedure in this representation was worked out by Anderson, Yuval and Hamann in the seminal paper [19]. To conform with established notation we introduce dimensionless coupling constants $\alpha_{L,R}$ such that the bath spectra are given by $\chi_K(\omega) = \pi\alpha_K\omega$. The RG flow equations for the one-bath spin-boson model[1,20] are directly applicable for several baths,

$$\frac{d(\Delta/\omega_c)}{d\ln\omega_c} = -(1-\alpha)\left(\frac{\Delta}{\omega_c}\right) \qquad (27)$$

$$\frac{d\alpha}{d\ln\omega_c} = \alpha\left(\frac{\Delta}{\omega_c}\right)^2, \qquad (28)$$

with $\alpha = \alpha_L + \alpha_R$ being a combined coupling for both baths. These equations are valid for $\alpha < 4$ and $\Delta/\omega_c \lesssim 1$. The flow of the parameters with decreasing ω_c is depicted in Fig. 5. There a three qualitatively different regimes which we call A, B, and C.

In regime A the renormalization of α is small in a realistic case where the bare value of Δ/ω_c is small, so we can treat α as a constant. Then solving Eq. (27) yields[1,20]

$$\Delta = \Delta_0 \left(\frac{\omega_c}{\omega_{c0}} \right)^\alpha, \qquad (29)$$

where Δ_0 and ω_{c0} are the initial, unrenormalized values of Δ and ω_c. The cutoff is taken all the way down to the temperature T, and we obtain $\Delta \propto T^\alpha$. Now we can again examine the T dependence of conductance. From Eq. (26),

$$G \propto \frac{\sin^4\theta}{\omega_0^2} T^3 = \frac{\Delta^4}{(\Delta^2 + \epsilon^2)^3} T^3 \propto \begin{cases} T^{3+4\alpha}, & \epsilon \gg \Delta \\ T^{3-2\alpha}, & \epsilon \ll \Delta \end{cases}$$
$$(30)$$

Thus the RG calculation gives a non-perturbative correction to the bare T^3 behavior depending on the interaction strength α. We have extracted two limiting forms of this correction, depending on the relative magnitudes of the TLS detuning and the renormalized tunneling element. Since Δ always renormalizes to zero, in the asymptotic low-temperature limit thermal conductance always obeys $G \propto T^{3+4\alpha}$ even if the bare parameters satisfy $\epsilon \ll \Delta$.

We would like to make a brief comment regarding the validity of the above considerations. There are two possible restrictions, namely the validity of flow equations, Eqs. (27) and (28), and the validity of the perturbation calculation for the current, Eq. (26). Since α and Δ/ω_c move towards smaller values, the flow equations remain valid throughout the process, all the way to $\omega_c \to 0$. Thus Eq. (30) can be used even in the zero temperature limit. On the other hand, validity of Eq. (26) requires that the perturbation expansion parameter, essentially $\alpha \sin^2\theta$, is small. In regime A, where $\alpha > 1$, this means that θ must be small, i.e., $\Delta/\epsilon \ll 1$, so the possibility $G \propto T^{3-2\alpha}$ is not realized in the region of validity of the perturbation theory and must be discarded.

In regime B we can also accurately approximate α with a constant value, and therefore Eq. (30) applies also to this case. However, now the flow takes Δ/ω_c towards larger values eventually leaving the region of validity of the flow equations. Therefore the flow should be terminated when $\Delta/\omega_c = 1$, and at this point the value of the cutoff is[1,20]

$$\omega_{c,\mathrm{min}} = \omega_{c0} \left(\frac{\Delta_0}{\omega_{c0}} \right)^{\frac{1}{1-\alpha}}. \qquad (31)$$

Thus Eq. (30) can only be used down to temperatures $T \sim \omega_{c,\mathrm{min}}$. The perturbation parameter $\alpha \sin^2\theta$ is always small for any θ by taking a small enough α, so both limiting cases of Eq. (30) are relevant above the cutoff temperature $\omega_{c,\mathrm{min}}$. In the asymptotic weak-coupling

limit $\alpha \to 0$ one recovers the simple $G \propto T^3$-dependence as expected by the perturbation calculation.

Regime C has the most complicated flow and α cannot be treated as a constant, prohibiting simple analytical estimates. Also in this case Δ/ω_c is a relevant perturbation eventually flowing to strong coupling. In principle the flow equations can still be used to analyze the temperature dependence of thermal conductance above the cutoff energy by numerically solving the flow.

V. CONCLUSIONS

We studied thermal transport in the spin-boson model at temperatures below the TLS energy splitting. At low temperatures a naive Golden-Rule approximation becomes invalid and transport is dominated by two-boson processes. Qualitatively the behavior is similar to electronic transport in quantum dots under Coulomb blockade when lowest-order processes are exponentially suppressed. Applying perturbation methods we obtained an analytical description of low-temperature transport and investigated the crossover between the low-temperature and high-temperature limits. The crossover region is sharp and an accurate analytical description at all temperatures is provided by patching the two limiting cases together. In the case of reservoirs with simple power law spectral densities $\chi_{L/R}(\omega) \propto \omega^{s_{L/R}}$ the low-temperature thermal conductance obeys $G(T) \propto T^{1+s_L+s_R}$. Going beyond perturbation theory, we applied renormalization group arguments to extract anomalous temperature scaling of thermal conductance in the Ohmic case. The power law predicted by simple perturbation theory was modified by interaction-dependent correction which becomes significant outside the weak-coupling regime.

The studied problem is not only important for future applications, but also relevant in systems available for experimental studies presently. One of the most flexible systems to probe our predictions is a superconducting flux qubit coupled to two linear circuits acting as reservoirs.[12,21] In this realization the electromagnetic fluctuations of the reservoirs play the role of bosonic modes in the spin-boson model. Using external fields to tune the energy splitting of the qubit it is possible to address different regimes in a controlled manner. Considering the generic role of the spin-boson model, it is expected that our results are important for a variety of applications in heattronics of molecular and nanoscale systems in the future.

Upon completion of the manuscript we learned about a new work by Wu and Segal.[22] They have studied a system similar to ours and their results agree with Eq. (25).

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